Math Analysis

2.5 part 2 Fundamental Theorem of Algebra, Descartes' Rule of Signs

The Fundamental Theorem of Algebra

- A polynomial has at least one solution in the set of complex numbers.
- The degree of a polynomial will tell you how many roots/zeros the function has.

EXAMPLE 1 **Find the number of solutions or zeros**

a. How many solutions does the equation $x^3 + 5x^2 + 4x + 20 = 0$ have? 3

b. How many zeros does the function $f(x) = x^4 - 8x^3 + 18x^2 - 27$ have? 4

Complex Conjugates Theorem

If a + bi is an imaginary zero/root then its conjugate a - bi is also a zero/root of the polynomial function/equation.

Irrational Conjugates Theorem

If a + $\sqrt{b}$ is a zero/root then so is its conjugate a - $\sqrt{b}$

EXAMPLE 2 Find the zeros of a polynomial function

Find all zeros of $f(x) = x^5 - 4x^4 + 4x^3 + 10x^2 - 13x - 14$. 2, −1, −1

$\frac{p}{q}$ $\frac{14}{1}$ ±1, ±2, ±7, ±14

$$\begin{array}{r|rrrrrr} 1 & 1 & -4 & 4 & 10 & -13 & -14 \\ & & 1 & -3 & 1 & 11 & -2 \\ \hline & 1 & -3 & 1 & 11 & -2 & \times \end{array}$$

$$\begin{array}{r|rrrrrr} 2 & 1 & -4 & 4 & 10 & -13 & -14 \\ & & 2 & -4 & 0 & 20 & 14 \\ \hline & 1 & -2 & 0 & 10 & 7 & 0 \checkmark \end{array}$$

$$\begin{array}{r|rrrrr} -2 & 1 & -2 & 0 & 10 & 7 \\ & & -2 & 8 & -16 & 12 \\ \hline & 1 & -4 & 8 & -6 & \times \end{array}$$

$$\begin{array}{r|rrrrr} -1 & 1 & -2 & 0 & 10 & 7 \\ & & -1 & 3 & -3 & -7 \\ \hline & 1 & -3 & 3 & 7 & 0 \checkmark \end{array}$$

$$\begin{array}{r|rrrr} 7 & 1 & -3 & 3 & 7 \\ & & 7 & 28 & \\ \hline & 1 & 4 & 31 & \times \end{array}$$

$$\begin{array}{r|rrrr} 14 & 1 & -3 & 3 & 7 \\ & & 14 & & \\ \hline & 1 & 11 & & \times \end{array}$$

$1x^3 - 3x^2 + 3x + 7$

$$\begin{array}{r|rrrr} -1 & 1 & -3 & 3 & 7 \\ + & & -1 & 4 & -7 \\ \hline & 1 & -4 & 7 & 0 \checkmark \end{array}$$

$x^2 - 4x + 7$

$$x = \frac{4 \pm \sqrt{16 - 4(7)}}{2}$$

$$\frac{4 \pm \sqrt{-12}}{2} \quad 3 \cdot 4 \quad \frac{4 \pm 2i\sqrt{3}}{2} \quad \div 2$$

$$\boxed{2, -1, -1, 2 \pm i\sqrt{3}}$$

Example: Finding a Polynomial Function with Given Zeros

Find a third-degree polynomial function $f(x)$ with real coefficients that has –3 and i as zeros and such that $f(1) = 8$.

$f(x) = y$

roots $-3,\ i,\ -i$

factors $a(x+3)(x-i)(x+i)$

$(x-i)(x+i)$

$x^2 \cancel{ix - ix} - i^2$

$-(-1)$

$(x^2+1)(x+3)$

$a_n(x^3+3x^2+x+3) = y$

$a_n(1^3+3(1)^2+1+3) = 8$

$1+3+4$

$a_n(8)$

$\frac{8a_n}{8} = \frac{8}{8}$

$a_n = 1$

$1(x^3+3x^2+x+3)$

$$\boxed{x^3+3x^2+x+3}$$

Try This:

Find all zeros of the polynomial function.

$$f(x) = x^5 - 2x^4 \overset{0x^3}{+} 8x^2 - 13x + 6 \quad -2, 1$$

$\frac{p}{q} = \pm 1, \pm 2, \pm 3, \pm 6$

$$\begin{array}{r|rrrrrr} -2 & 1 & -2 & 0 & 8 & -13 & 6 \\ + & \downarrow & -2 & 8 & -16 & 16 & -6 \\ \hline & 1 & -4 & 8 & -8 & 3 & 0 \checkmark \end{array}$$

$$\begin{array}{r|rrrrr} 1 & 1 & -4 & 8 & -8 & 3 \\ & & 1 & -3 & 5 & -3 \\ \hline & 1 & -3 & 5 & -3 & 0 \checkmark \end{array}$$

$$\begin{array}{r|rrrr} 1 & 1 & -3 & 5 & -3 \\ & & 1 & -2 & 3 \\ \hline & 1 & -2 & 3 & 0 \checkmark \end{array}$$

$x^2 - 2x + 3$

$\frac{2 \pm \sqrt{4-12}}{2}$

$\frac{2 \pm \sqrt{-8}}{2}$ $\sqrt{4 \cdot 2}$

$\frac{2 \pm 2i\sqrt{2}}{2}$

$\boxed{x = 1 \pm i\sqrt{2}, 1, 1, -2}$

Write a polynomial with the given information
n = 4; -2, 5, and 3 + 2i are zeros; f(1) = -96

$(x+2)(x-5)(x-3+2i)(x-3-2i)$

$(x^2-3x-10)(x^2-3x+2ix-3x+9-6i$
$-2ix+6i-4i^2)$
(-1)

$(x^2-3x-10)(x^2-6x+13)$

$x^4-6x^3+13x^2-3x^3+18x^2-39x-10x^2$
$+60x-130$

$a_n(x^4-9x^3+21x^2+21x-130)$

$a_n(1^4-9(1)^3+21(1)^2+21(1)-130) = -96$

$a_n(1-9+21+21-130) = -96$

$a_n(-96) = -96$

$a_n = 1$

$1(x^4-9x^3+21x^2+21x-130)$

$\boxed{x^4-9x^3+21x^2+21x-130}$

Descartes' Rule of Signs

The number of **positive real zeros** is the same as the number of changes in the sign of the coefficients of the terms of P(x) or less than that by an even number.

The number of **negative real zeros** is the same as the number of changes in the sign of the coefficients of the terms of P(-x) or less than that by an even number.

Ex) Find the number of possible positive and negative real zeros for $f(x) = 2x^5 + 3x^4 - 6x^3 + 6x^2 - 8x + 3$. Then determine the zeros.

+ + − + − +

1 2 3 4

Pos real zeros: $\boxed{4 \text{ or } 2 \text{ or } 0}$

Neg real zeros:

$P(-x) = 2(-x)^5 + 3(-x)^4 - 6(-x)^3 + 6(-x)^2 - 8(-x) + 3$

$P(-x) = -2x^5 + 3x^4 + 6x^3 + 6x + 8x + 3$

− + + + + +

1

Ex) Find the number of possible positive and negative real zeros for $f(x) = 2x^4 - 3x^3 - 2x^2 + 5x + 1$. Then determine the zeros.

1 0 1 0

Pos Real Zeros : 2 or 0

Neg Real Zeros : 2 or 0

$f(-x) = 2x^4 + 3x^3 - 2x^2 - 5x + 1$

0 1 0 1

Homework

pg 357; 26-46e