College Algebra Prerequisite Topics Review

- Quick review of basic algebra skills that you should have developed before taking this class
- 18 problems that are typical of things you should already know how to do

Review of Like Terms

- Recall that a **term** is a constant, a variable, or a product of a constant and variables
- **Like Terms:** terms are called "like terms" if they have exactly the same variables with exactly the same exponents, but <u>may</u> have different coefficients
- Example of Like Terms:

$$\boxed{3x^2y \quad and \quad -7x^2y}$$

Adding and Subtracting Like Terms

- When "like terms" are added or subtracted, the result is a like term and its coefficient is the sum or difference of the coefficients of the other terms
- Examples:

$$-2x+7x-x=\boxed{4x}$$

$$4x^2-19x^2y+6xy+2x^2-x^2y=\boxed{6x^2-20x^2y+6xy}$$

Polynomial

- **Polynomial** – a finite sum of terms
- Examples:

$$6x^2 - 5x - 4$$

How many terms? $\boxed{3}$ (Trinomial)

Degree of first term? $\boxed{2}$

Coefficient of second term? $\boxed{-5}$

$$3x^2y - 5x^4y^6$$

How many terms? $\boxed{2}$ (Binomial)

Degree of second term? $\boxed{10}$

Coefficient of first term? $\boxed{3}$

Adding and Subtracting Polynomials

- To add or subtract polynomials:
  - Distribute to get rid of parentheses
  - Combine like terms
- Example:

$$\left(2x^2 - 3x + 1\right) + \left(x^2 + x - 3\right) - \left(3x - 2\right)$$

$$2x^2 - 3x + 1 + x^2 + x - 3 - 3x + 2$$

$$\boxed{3x^2 - 5x}$$

Problem 1

- Perform the indicated operation:

$$\left(6x^4 - 3x^2 + x\right) - \left(2x^3 + 5x^2 + 4x\right) + \left(x^3 - x\right)$$

- Answer:

$$6x^4 - x^3 - 8x^2 - 4x$$

Multiplying Polynomials

- To multiply polynomials:
  - Get rid of parentheses by multiplying every term of the first by every term of the second using the rules of exponents
  - Combine like terms
- Examples:

$$(x+3)(2x^2-5x+4) = 2x^3-5x^2+4x+6x^2-15x+12 = \boxed{2x^3+x^2-11x+12}$$

$$(2x+3)(5x-4) = 10x^2-8x+15x-12 = \boxed{10x^2+7x-12}$$

Problem 2

- Perform the indicated operation:

$$(2x-1)\left(x^2-3x+4\right)$$

- Answer:

$$2x^3-7x^2+11x-4$$

Squaring a Binomial

- To square a binomial means to multiply it by itself

$$(2x+3)^2 = (2x+3)(2x+3) = 4x^2+6x+6x+9 = \boxed{4x^2+12x+9}$$

- Although a binomial can be squared by foiling it by itself, it is best to memorize a shortcut for squaring a binomial:

$$(a+b)^2 = a^2+2ab+b^2 \qquad \boxed{\text{first}^2+2(\text{first})(\text{second})+\text{second}^2}$$

$$(2x+3)^2 = \boxed{4x^2+12x+9}$$

Problem 3

- Perform the indicated operation:

$$(x+3)^2$$

- Answer:

$$x^2+6x+9$$

Dividing a Polynomial by a Polynomial

- First **write each** polynomial **in descending powers**
- **If** a **term** of some power **is missing**, **write that term with** a **zero coefficient**
- **Complete** the problem exactly **like** a **long division** problem **in basic math**

Example

$$\left(-2x^2+3x^3-150\right)\div\left(x^2-4\right)$$

$$\left(3x^3-2x^2+0x-150\right)\div\left(x^2+0x-4\right)$$

$$3x \quad -2 \quad +\frac{12x-158}{x^2-4}$$

$$x^2+0x-4\ \overline{\big)\,3x^3-2x^2+0x-150}$$

$$-\left(\underline{3x^3+0x^2-12x}\right)$$

$$-2x^2+12x-150$$

$$-\left(\underline{-2x^2-\ 0x+\ \ 8}\right)$$

$$12x-158$$

Problem 4

- Perform the indicated operation:

$$\frac{6x^3 - x + 7x^2 + 2}{3x + 2}$$

- Answer:

$$2x^2 + x - 1 + \frac{4}{3x + 2}$$

Factoring Polynomials

- To **factor a polynomial** is to **write** it **as** a **product of two or more other polynomials**, each of which is called a factor
- In a sense, **factoring is** the **opposite of multiplying** polynomials:

We have learned that:

$\mathbf{(2x - 3)(3x + 5) = 6x^2 + x - 15}$

If we were asked to factor $\mathbf{6x^2 + x - 15}$ we would write it as:

$\mathbf{(2x - 3)(3x + 5)}$

So we would say that $\mathbf{(2x - 3)}$ and $\mathbf{(3x + 5)}$ are **factors of** $\mathbf{6x^2 + x - 15}$

Prime Polynomials

- A **polynomial** is called **prime**, **if** it is not 1, and if its **only factors are itself and 1**
- Just like we learn to identify certain numbers as being prime **we will learn to identify** certain **polynomials as being prime**
- We **will also completely factor polynomials** by writing them **as a product of prime polynomials**

Importance of Factoring

- If you don’t learn to factor polynomials you can’t pass college algebra or more advanced math classes
- It is essential that you memorize the following procedures and become proficient in using them

5 Steps in Completely Factoring a Polynomial

(1) Write the polynomial in descending powers of one variable (if there is more than one variable, pick any one you wish)

(2) Look at each term of the polynomial to see if every term contains a common factor other than 1, if so, use the distributive property in reverse to place the greatest common factor outside a parentheses and other terms inside parentheses that give a product equal to the original polynomial

(3) After factoring out the greatest common factor, look at the new polynomial factors to determine how many terms each one contains

(4) Use the method appropriate to the number of terms in the polynomial:

4 or more terms: **"Factor by Grouping"**

3 terms: **PRIME UNLESS** they are of the form "$ax^2 + bx + c$". If of this form, use **"Trial and Error FOIL"** or **"abc Grouping"**

2 terms: Always **PRIME UNLESS** they are:

"difference of squares":	$a^2 – b^2$
"difference of cubes":	$a^3 – b^3$
"sum of cubes":	$a^3 + b^3$

In each of these cases factor by a formula

(5) Cycle through step 4 as many times as necessary until all factors are "prime"

Factoring the Greatest Common Factor from Polynomials

(Already in descending powers of a variable)

$9y^5 + y^2$ What is the GCF? y^2

$y^2(\qquad\qquad)$

$y^2(9y^3 + 1)$

$6x^2t + 8xt + 12t$ What is the GCF? $2t$

$2t(\qquad\qquad\qquad)$

$2t(3x^2 + 4x + 6)$

Factor by Grouping

(Used for 4 or more terms)

(1) Group the terms by underlining:

If there are exactly 4 terms try:

2 & 2 grouping, 3 & 1 grouping, or 1 & 3 grouping

If there are exactly 5 terms try:

3 & 2 grouping, or 2 & 3 grouping

Factoring by Grouping

(2) Factor each underlined group as if it were a factoring problem by itself

(3) Now determine if the underlined and factored groups contain a common factor,

if they contain a common factor, factor it out

if they don't contain a common factor, try other groupings, if none work, the polynomial is prime

(4) Once again count the terms in each of the new polynomial factors and return to step 4.

Example of Factoring by Grouping

Factor: ax + ay + 6x + 6y

(1) Group the terms by underlining (start with 2 and 2 grouping):

$\underline{ax + ay} + \underline{6x + 6y}$

(2) Factor each underlined group as if it were a factoring problem by itself:

$\underline{a(x + y)} + \underline{6(x + y)}$

[notice sign between groups gets carried down]

Factoring by Grouping Example Continued

(3) Now determine if the underlined and factored groups contain a common factor, if they do, factor it out:

<u>a(x + y)</u> + <u>6(x + y)</u>

(x + y)(a + 6)

ax + ay + 6x + 6y = (x + y)(a + 6) Now factored

(4) Once again count the terms in each of the new polynomial factors and return to step 4.

Each of these polynomial factors contains two terms, return to step 4 to see if these will factor (SINCE WE HAVE NOT YET DISCUSSED FACTORING POLYNOMIALS WITH TWO TERMS WE WILL NOT CONTINUE AT THIS TIME)

Example of Factoring by Grouping

Factor: $2x^2 + 3x - 2xy - 3y$

(1) Group the terms by underlining (Try 2 and 2 grouping):

$$\underline{2x^2 + 3x} - \underline{2xy - 3y}$$

(2) Factor each underlined group as if it were a factoring problem by itself:

$$\underline{x(2x+3)} - \underline{y(2x+3)}$$

[notice sign between groups gets carried down and you have to be careful with this sign]

Factoring by Grouping Example Continued

(3) Now determine if the underlined and factored groups contain a common factor, if they do, factor it out:

$$x\underline{(2x+3)} - y\underline{(2x+3)}$$

$$(2x+3)(x-y) \qquad \text{Now factored}$$

(4) Once again count the terms in each of the new polynomial factors and return to step 4.

Each of these polynomial factors contains two terms, return to step 4 to see if these will factor (AGAIN WE HAVE LEARNED TO FACTOR BINOMIALS YET, SO WE WON'T CONTINUE ON THIS EXAMPLE)

Note on Factoring by Grouping

- It was noted in step 3 of the factor by grouping steps that sometimes the first grouping, or the first arrangement of terms might not result in giving a common factor in each term – in that case other groupings, or other arrangements of terms, must be tried
- Only after we have tried all groupings and all arrangement of terms can we determine whether the polynomial is factorable or prime

Try Factoring by Grouping Without First Rearranging

Factor: $12x - 3y + 9xy - 4$

(1) Group the terms by underlining (Try 2 and 2):

$$\underline{12x - 3y} + \underline{9xy - 4}$$

(2) Factor each underlined group as if it were a factoring problem by itself:

$$\underline{3(\qquad\qquad)} + \underline{1(\qquad\qquad)}$$

$$3(4x - y) + 1(9xy - 4)$$ What's the problem with trying to continue?

No common factor in the two underlined groups!

Now Try Same Problem by Rearranging

Factor: $12x - 3y + 9xy - 4$

Rearrange: $9xy + 12x - 3y - 4$

(1) Group the terms by underlining:

$$\underline{9xy + 12x} - \underline{3y - 4}$$

(2) Factor each underlined group as if it were a factoring problem by itself:

$$\underline{3x(\qquad)} - \underline{1(\qquad)}$$

$$\underline{3x(3y+4)} - \underline{1(3y+4)}$$

Can we continue factoring now?

Yes, there is a common factor in the two underlined groups!

Factoring by Grouping Example Continued

(3) Now factor out the common factor:

$$3x\underline{(3y+4)}-1\underline{(3y+4)}$$

$$(3y+4)(\qquad)$$

$$(3y+4)(3x-1) \quad \text{Rearranging made factoring possible!}$$

$$\text{DOESN'T ALWAYS HELP!}$$

(4) Once again count the terms in each of the new polynomial factors and return to step 4.

Each of these polynomial factors contains two terms, return to step 4 to see if these will factor (AGAIN WE TO WAIT UNTIL WE LEARN TO FACTOR BINOMIALS BEFORE WE CAN CONTINUE)

Factoring Trinomials by

Trial and Error FOIL

(Used for 3 terms of form $ax^2 + bx + c$)

- Given a trinomial if this form, experiment to try to find two binomials that could multiply to give that trinomial
- Remember that when two binomials are multiplied:

First times First = First Term of Trinomial

Outside times Outside + Inside times Inside = Middle Term of Trinomial

Last times Last = Last Term of Trinomial

Steps in Using Trial and Error FOIL

- Given a trinomial of the form:

$$\underline{ax}^2 + bx + c$$

- Write two blank parentheses that will each eventually contain a binomial

$$(\_\_\ \ \_\_)(\_\_\ \ \_\_)$$

- Use the idea that “first times first = first” to get possible answers for first term of each binomial

$$(\_\_\quad\ )(\_\_\quad\ )$$

Continuing Steps in Trial and Error FOIL

- Given a trinomial of the form:

$$ax^2 + bx + \underline{c}$$

- Next use the idea that “last times last = last” to get possible answers for last term of each binomial

$$(\quad \underline{\quad})(\quad \underline{\quad})$$

Continuing Steps in Trial and Error FOIL

- Given a trinomial of the form:

$$ax^2 + bx + c$$

- Finally use the idea that "Outside times Outside + Inside times Inside = Middle Term of Trinomial" to get the final answer for two binomials that multiply to give the trinomial

$$(\_\_\ \ \_\_)(\_\_\ \ \_\_)$$

Prime Trinomials

- A trinomial is automatically prime if it is not of the form: $ax^2 + bx + c$
- However, a trinomial of this form is also prime if **<u>all</u>** possible combinations of "trial and error FOIL" have been tried, and none have yielded the correct middle term
- Example: Why is this prime? $x^2 + 5x - 3$
- The only possible combinations that give the correct first and last terms are:

$$(x-3)(x+1) \quad and \quad (x+3)(x-1)$$

- Neither gives the correct middle term:

$$x^2 - 2x - 3 \quad and \quad x^2 + 2x - 3$$

Example of Factoring by Trial and Error FOIL

- Factor: **$12x^2 + 11x – 5$**
- Using steps on previous slides, we see all the possibilities that give the correct first and last terms on the left and the result of multiplying them on the right (we are looking for the one that gives the correct middle term):

$(12x + 1)(x – 5) = 12x^2 – 59x – 5$
$(12x – 1)(x + 5) = 12x^2 + 59x – 5$
$(12x + 5)(x – 1) = 12x^2 – 7x – 5$
$(12x – 5)(x + 1) = 12x^2 + 7x – 5$
$(6x + 1)(2x – 5) = 12x^2 – 28x – 5$
$(6x – 1)(2x + 5) = 12x^2 +28x – 5$
$(6x + 5)(2x – 1) = 12x^2 + 4x – 5$
$(6x – 5)(2x + 1) = 12x^2 – 4x – 5$
$(4x + 1)(3x – 5) = 12x^2 – 17x – 5$
$(4x – 1)(3x + 1) = 12x^2 + x – 5$
$(4x + 5)(3x – 1) = 12x^2 +11x – 5$
$(4x – 5)(3x + 1) = 12x^2 -11x – 5$

Only Correct Factoring

A Second Method of Factoring Trinomials

- While the “Trial and Error FOIL” method can always be used in attempting to factor trinomials, and is usually best when first and last terms have “small coefficients,” there is a second method that is usually best to use when first and last coefficients are “larger”
- We call the second method: “abc grouping”

Factoring Trinomials by
abc Grouping
(Used for 3 terms of form $ax^2 + bx + c$)

- When a polynomial is of this form:

  $ax^2 + bx + c$

(1) Identify “a”, “b”, and “c”

(2) Multiply “a” and “c”

(3) Find two numbers “m” and “n”, that multiply to give “ac” and add to give “b” (If this can not be done, the polynomial is already prime)

(4) Rewrite polynomial as: $ax^2 + mx + nx + c$

(5) Factor these four terms by 2 and 2 grouping

Example of Factoring by abc Grouping

$$ac = -60$$

$$\underline{15} \times \underline{-4}$$

$$m + n = 11$$

- Factor: $\mathbf{12x^2 + 11x - 5}$

(1) Identify “a”, “b”, and “c”
a = 12, b = 11, c = - 5

(2) Multiply “a” and “c”
ac = - 60

(3) Find two numbers “m” and “n”, that multiply to give “ac” and add to give “b” (If this can not be done, the polynomial is already prime)
m = 15 and **n = - 4**, because **mn = -60** and **m + n = 11**

(4) Rewrite as four terms: $\mathbf{12x^2 + 15x - 4x - 5}$

(5) Factor by grouping:
$\mathbf{\underline{12x^2 + 15x} - \underline{4x - 5}}$
$\mathbf{\underline{3x(4x + 5)} - \underline{1(4x + 5)}}$
$\mathbf{(4x + 5)(3x - 1)}$

Example of Factoring by **abc Grouping** (with two variables)

- Factor: $\mathbf{35x^2 – 12y^2 – 13xy}$

  $\mathbf{35x^2 – 13xy – 12y^2}$ (descending powers of x)

(1) Identify "a", "b", and "c" (Ignore y variable)

a = 35, b = - 13, c = - 12

(2) Multiply "a" and "c"

ac = - 420

(3) Find two numbers "m" and "n", that multiply to give "ac" and add to give "b" (If this can not be done, the polynomial is already prime)

m = 15 and **n = - 28**, because mn = - 420 and m + n = - 13

(4) Rewrite as four terms: $\mathbf{35x^2 + 15xy – 28xy – 12y^2}$

(5) Factor by grouping: $\underline{\mathbf{35x^2 + 15xy}}$ **–** $\underline{\mathbf{28xy – 12y^2}}$

$\underline{\mathbf{5x(7x + 3y)}}$ **–** $\underline{\mathbf{4y(7x + 3y)}}$

$\mathbf{(7x + 3y)(5x – 4y)}$

Factoring Binomials by Formula

- Factor by using formula appropriate for the binomial:

"difference of squares":

$$a^2 - b^2 = (a - b)(a + b)$$

"difference of cubes":

$$a^3 - b^3 = (a - b)(a^2 + ab + b^2) \quad \textbf{Trinomial is prime}$$

"sum of cubes":

$$a^3 + b^3 = (a + b)(a^2 - ab + b^2) \quad \textbf{Trinomial is prime}$$

- **If none of the formulas apply, the binomial is prime**
  BINOMIALS ARE PRIME UNLESS THEY ARE ONE OF THESE

Example of Factoring Binomials

- Factor: $\mathbf{25x^2 - 9y^2}$
- Note that this binomial is a difference of squares:

  $(5x)^2 - (3y)^2$
- Using formula gives:

  $\mathbf{(5x - 3y)(5x + 3y)}$

Example of Factoring Binomials

- Factor: $\mathbf{8x^3 - 27}$
- Note that this is a difference of cubes:

  $(2x)^3 - (3)^3$
- Using formula gives:

  $\mathbf{(2x - 3)(4x^2 + 6x + 9)}$

Example of Factoring Binomials

- Factor: **$4x^2 + 9$**
- Note that this is not a difference of squares, difference of cubes, or sum of cubes, therefore it is **prime**
- **$(4x^2 + 9)$**
- To show factoring of a polynomial that is prime, put it inside parentheses

Problem 5

- Factor completely:

$$x^2 - 3x - 4$$

- Answer: $(x-4)(x+1)$

Problem 6

- Factor completely:

$$3x^2 + 12$$

- Answer: $3\left(x^2 + 4\right)$

Problem 7

- Factor completely:

  $x^2 - 9$

- Answer: $(x-3)(x+3)$

Problem 8

- Factor completely:

  $x^3 + 8$

- Answer: $(x+2)(x^2 - 2x + 4)$

Problem 9

- Factor completely:

$$12x + 4x^3 - 2x^2 - 6$$

- Answer: $2(2x-1)(x^2+3)$

Rational Expression

- A **ratio of two polynomials** where the denominator is not zero (an **"ugly fraction"** with a **variable in** a **denominator**)
- Example:

$$\frac{x^2 - x - 2}{x + 3}$$

Reducing Rational Expressions to Lowest Terms

- **Completely factor** both **numerator** and **denominator**
- Apply the fundamental principle of fractions: **divide out common factors** that are found in both the numerator and the denominator

Example of Reducing Rational Expressions to Lowest Terms

- Reduce to lowest terms: $$\frac{3x^3+24}{3x+6}$$
- Factor top and bottom: $$\frac{3\left(x^3+8\right)}{3\left(x+2\right)}$$
$$\frac{\overset{1}{\cancel{3}}\overset{1}{\cancel{\left(x+2\right)}}\left(x^2-2x+4\right)}{\underset{1}{\cancel{3}}\underset{1}{\cancel{\left(x+2\right)}}}$$
- Divide out common factors to get:

$$x^2-2x+4$$

Example of Reducing Rational Expressions to Lowest Terms

- Reduce to lowest terms: $$\frac{x-3}{3-x}$$
- Factor top and bottom: $$\frac{x-3}{-x+3} = \frac{1(x-3)}{-1(x-3)}$$

$$\frac{1\cancel{(x-3)}}{-1\cancel{(x-3)}} =$$

$$1$$

- Divide out common factors to get:

$$-1$$

Problem 10

- Reduce to lowest terms:

$$\frac{x^2 + 3x}{x^2 + 2x - 3}$$

- Answer: $\dfrac{x}{x-1}$

Finding the Least Common Denominator, LCD, of Rational Expressions

- Completely **factor each denominator**
- Construct the **LCD** by writing down **each factor** the **maximum** number of **times** it is found **in any denominator**

Example of Finding the LCD

- Given three denominators, find the LCD:

$$3x^2 - 12 \quad , \qquad 6x - 12 \quad , \qquad 4x^2 - 16x + 16$$

- Factor each denominator:

$$3x^2 - 12 = 3\left(x^2 - 4\right) = 3\left(x - 2\right)\left(x + 2\right)$$

$$6x - 12 = 6\left(x - 2\right) = 2 \cdot 3\left(x - 2\right)$$

$$4x^2 - 16x + 16 = 4\left(x^2 - 4x + 4\right) = 2 \cdot 2\left(x - 2\right)\left(x - 2\right)$$

- Construct LCD by writing each factor the maximum number of times it's found in any denominator:

$$\text{LCD} = 2 \cdot 2 \cdot 3\left(x - 2\right)\left(x - 2\right)\left(x + 2\right)$$

$$\text{LCD} = 12\left(x - 2\right)^2\left(x + 2\right)$$

Adding and Subtracting Rational Expressions (Same as Fractions)

- **Find** a least common denominator, **LCD**, for the rational expressions
- **Write each** fraction **as** an **equivalent fraction having** the **LCD**
- **Write** the **answer by adding or subtracting numerators** as indicated, **and keeping** the **LCD**
- If possible, **reduce** the answer **to lowest terms**

Example

$$\frac{y+2}{y^2-y}-\frac{3y}{2y^2-4y+2}+\frac{1}{y} \quad =\frac{y+2}{y(y-1)}-\frac{3y}{2(y-1)(y-1)}+\frac{1}{y}$$

- Find a least common denominator, LCD, for the rational expressions:

$$y(y-1)$$
$$2(y-1)(y-1)$$
$$y$$

$$\boxed{\begin{array}{c}\text{LCD} = \\ 2y(y-1)(y-1)\end{array}}$$

- Write each fraction as an equivalent fraction having the LCD:

$$\frac{2(y+2)(y-1)}{2y(y-1)(y-1)}-\frac{3y\cdot y}{2y(y-1)(y-1)}+\frac{2\cdot 1(y-1)(y-1)}{2y(y-1)(y-1)}$$

- Write the answer by adding or subtracting numerators as indicated, and keeping the LCD:

$$=\frac{2(y^2+y-2)-3y^2+2(y^2-2y+1)}{2y(y-1)(y-1)}=\frac{2y^2+2y-4-3y^2+2y^2-4y+2}{2y(y-1)(y-1)}=$$

- If possible, reduce the answer to lowest terms

$$\frac{y^2-2y-2}{2y(y-1)(y-1)} \quad \text{Since top won't factor, fraction won't reduce!}$$

Problem 11

- Subtract and reduce to lowest terms:

$$\frac{x}{x-1} - \frac{1}{x^2 - x}$$

- Answer: $\frac{x+1}{x}$

Multiplying Rational Expressions (Same as Multiplying Fractions)

- **Factor** each **numerator** and **denominator**
- **Divide out common factors**
- **Write answer** (leave polynomials in factored form)
- Example:

$$\frac{4}{9}\cdot\frac{15}{28} = \frac{\overset{1}{\cancel{2}}\cdot\overset{1}{\cancel{2}}}{3\cdot\underset{1}{\cancel{3}}}\cdot\frac{\overset{1}{\cancel{3}}\cdot 5}{\underset{1}{\cancel{2}}\cdot\underset{1}{\cancel{2}}\cdot 7} = \frac{5}{21}$$

Example of Multiplying Rational Expressions

$$\frac{3x^2-2x-8}{3x^2+14x+8}\cdot\frac{3x+2}{3x+4}$$

Completely factor each top and bottom:

$$\frac{\overset{1}{\cancel{(3x+4)}}(x-2)}{\underset{1}{\cancel{(3x+2)}}(x+4)}\cdot\frac{\overset{1}{\cancel{(3x+2)}}}{\underset{1}{\cancel{(3x+4)}}}$$

Divide out common factors:

$$\frac{(x-2)}{(x+4)}$$

Dividing Rational Expressions (Same as Dividing Fractions)

- **Invert** the **divisor** and **change** problem **to multiplication**

$$\frac{a}{b} \div \frac{c}{d} = \quad \frac{a}{b} \cdot \frac{d}{c} = \quad \frac{ad}{bc}$$

- Example:

$$\frac{2}{3} \div \frac{3}{4} = \quad \frac{2}{3} \cdot \frac{4}{3} = \quad \frac{8}{9}$$

Example of Dividing Rational Expressions

$$\frac{2y^2}{9} \div \frac{8y^5 + 4y^3}{27} = \frac{2y^2}{9} \cdot \frac{27}{8y^5 + 4y^3} = \frac{\overset{1}{\cancel{2}}\overset{1}{\cancel{y^2}}}{\underset{1}{\cancel{9}}} \cdot \frac{\overset{3}{\cancel{27}}}{\underset{2}{\cancel{4}}\underset{y}{\cancel{y^3}}\left(2y^2 + 1\right)} =$$

$$\frac{3}{2y\left(2y^2 + 1\right)}$$

Problem 12

- Divide and reduce to lowest terms:

$$\frac{x^2+x}{8x^2} \div \frac{x^2-1}{4}$$

- Answer:

$$\frac{1}{2x(x-1)}$$

Exponential Expression

- An exponential expression is: a^n
  where a is called the base and n is called the exponent
- An exponent applies only to what it is immediately adjacent to (what it touches)
- Example:

$3x^2$ Exponent applies only to x, not to 3

$-m^4$ Exponent applies only to m, not to negative

$(2x)^3$ Exponent applies to (2x)

Meaning of Exponent

- The meaning of an exponent depends on the type of number it is
- An exponent that is a natural number (1, 2, 3,…) tells how many times to multiply the base by itself
- Examples: $3x^2 = 3 \cdot x \cdot x$

$$-m^4 = -1 \cdot m \cdot m \cdot m \cdot m$$

$$(2x)^3 = (2x) \cdot (2x) \cdot (2x) = 8x^3$$

In the next section we will learn the meaning of any integer exponent

Rules of Exponents

- **Product Rule**: When two exponential expressions with the same base are multiplied, the result is an exponential expression with the same base having an exponent equal to the sum of the two exponents

$$\boxed{a^m \cdot a^n = a^{m+n}}$$

- Examples:

$$3^4 \cdot 3^2 = 3^{4+2} = 3^6$$
$$x^7 \cdot x^4 = x^{7+4} = x^{11}$$

Rules of Exponents

- **Power of a Power Rule**: When an exponential expression is raised to a power, the result is an exponential expression with the same base having an exponent equal to the product of the two exponents

$$\boxed{\left(a^{m}\right)^{n} = a^{mn}}$$

- Examples:

$$\left(3^{4}\right)^{2} = 3^{4\cdot 2} = 3^{8}$$

$$\left(x^{7}\right)^{4} = x^{7\cdot 4} = x^{28}$$

Rules of Exponents

- **Power of a Product Rule**: When a product is raised to a power, the result is the product of each factor raised to the power

$$\boxed{(ab)^n = a^n b^n}$$

- Examples:

$$(3x)^2 = 3^2 x^2 = 9x^2$$

$$(2y)^4 = 2^4 y^4 = 16y^4$$

Rules of Exponents

- **Power of a Quotient Rule**: When a quotient is raised to a power, the result is the quotient of the numerator to the power and the denominator to the power

$$\boxed{\left(\frac{a}{b}\right)^n = \frac{a^n}{b^n}}$$

- Example: $\left(\frac{3}{x}\right)^2 = \frac{3^2}{x^2} = \frac{9}{x^2}$

Using Combinations of Rules to Simplify Expression with Exponents

- Examples:

$$5\left(2m^2p^3\right)^4 = 5\cdot 2^4 m^8 p^{12} = 5\cdot 16 m^8 p^{12} = \boxed{80m^8p^{12}}$$

$$\left(-5x^2y^3\right)^3 = \left(-5\right)^3\cdot x^6y^9 = \boxed{-125x^6y^9}$$

$$\left(2x^2y^3\right)^3\left(-3x^2y^3\right)^2 = 8x^6y^9\cdot 9x^4y^6 = \boxed{72x^{10}y^{15}}$$

$$\frac{\left(2x^2y^3\right)^3}{\left(-3x^2y^5\right)^2} = \frac{8x^6y^9}{9x^4y^{10}} = \boxed{\frac{8x^2}{9y}}$$

Integer Exponents

- Thus far we have discussed the meaning of an exponent when it is a natural (counting) number: 1, 2, 3, …
- An exponent of this type tells us how many times to multiply the base by itself
- Next we will learn the meaning of zero and negative integer exponents
- Examples: 5^0

$$2^{-3}$$

Definition of Integer Exponents

- The patterns on the previous slide suggest the following definitions: $a^0 = 1$

$$a^{-n} = \left(\frac{1}{a}\right)^n$$

- These definitions work for any base, a , that is not zero:

$$5^0 = \boxed{1} \qquad 2^{-3} = \left(\frac{1}{2}\right)^3 = \boxed{\frac{1}{8}}$$

Quotient Rule for Exponential Expressions

- When exponential expressions with the <u>same</u> <u>base</u> are divided, the result is an exponential expression with the same base and an exponent equal to the numerator exponent minus the denominator exponent

$$\frac{a^m}{a^n} = a^{m-n}$$

Examples:

$$\frac{5^4}{5^7} = \quad 5^{4-7} = 5^{-3}$$

.

$$\frac{x^{12}}{x^4} = \quad x^{12-4} = x^8$$

"Slide Rule" for Exponential Expressions

- When **both** the **numerator** and **denominator** of a fraction are **factored** then **any factor may slide** from the top to bottom, or vice versa, by changing the sign on the exponent

  Example: Use rule to slide all factors to other part of the fraction:

$$\frac{a^m b^{-n}}{c^{-r} d^s} = \frac{c^r d^{-s}}{a^{-m} b^n}$$

- This rule applies to **all types of exponents**
- Often used to make all exponents positive

Simplify the Expression:

(Show answer with positive exponents)

$$\frac{8\left(y^3 y\right)^{-2}}{2^{-1} y^4 y^{-1}} = \frac{8y^{-6} y^{-2}}{2^{-1} y^4 y^{-1}} = \frac{8y^{-8}}{2^{-1} y^3} = \frac{8 \cdot 2^1}{y^3 y^8} = \frac{16}{y^{11}}$$

Problem 13

- Simplify and use only positive exponents in final answer:

$$\frac{\left(x^{-2}y^{1/3}\right)^{5}\left(8x^{2}y^{-2}\right)^{-2/3}}{x^{-3}y^{4}}$$

- Answer: $\dfrac{1}{4x^{25/3}y}$

Radical Notation

- **Roots** of real numbers may be indicated by means of either **rational exponent notation** or **radical notation**:

$$\sqrt[n]{a} \text{ is called a RADICAL (expression)}$$

$$\sqrt{} \text{ is called a RADICAL SIGN}$$

$$n \text{ is called the INDEX}$$

$$a \text{ is called the RADICAND}$$

Product Rule for Radicals

- When two radicals are <u>multiplied</u> that have the <u>same index</u> they may be combined as a single radical having that index and radicand equal to the product of the two radicands:

$$\sqrt[n]{a}\sqrt[n]{b} = \sqrt[n]{ab} \quad \Big| \quad \sqrt[4]{3}\sqrt[4]{5} = \sqrt[4]{3 \cdot 5} = \sqrt[4]{15}$$

- This rule works both directions:

$$\sqrt[n]{ab} = \sqrt[n]{a}\sqrt[n]{b} \quad \Big| \quad \sqrt[3]{16} = \sqrt[3]{8}\sqrt[3]{2} = 2\sqrt[3]{2}$$

Quotient Rule for Radicals

- When two radicals are <u>divided</u> that have the <u>same index</u> they may be combined as a single radical having that index and radicand equal to the quotient of the two radicands

$$\frac{\sqrt[n]{a}}{\sqrt[n]{b}} = \sqrt[n]{\frac{a}{b}} \qquad \frac{\sqrt[4]{5}}{\sqrt[4]{7}} = \sqrt[4]{\frac{5}{7}}$$

- This rule works both directions:

$$\sqrt[n]{\frac{a}{b}} = \frac{\sqrt[n]{a}}{\sqrt[n]{b}} \qquad \sqrt[3]{\frac{5}{8}} = \frac{\sqrt[3]{5}}{\sqrt[3]{8}} = \frac{\sqrt[3]{5}}{2}$$

.

Root of a Root Rule for Radicals

- When you take the m^th root of the n^th root of a radicand "a", it is the same as taking a single root of "a" using an index of "mn"

$$\sqrt[m]{\sqrt[n]{a}} = \sqrt[mn]{a}$$

$$\sqrt[4]{\sqrt[3]{6}} = \sqrt[12]{6}$$

.

Simplifying Radicals

- A radical **must be simplified** if any of the following conditions exist:

1. Some **factor of** the **radicand** has an **exponent** that is **bigger than** or **equal to** the **index**
2. There is a **radical in** a **denominator** (denominator needs to be “rationalized”)
3. The **radicand is a fraction**
4. **<u>All</u>** of the **factors of the radicand** have **exponents** that **share a common factor** with the **index**

Simplifying when Radicand has Exponent Too Big

$$\sqrt[3]{2^4}$$

1. Use the product rule to **write** the **single radical as a product of two radicals** where the **first** radicand **contains all factors whose exponents match** the **index** and the second radicand contains all other factors

$$\sqrt[3]{2^3}\sqrt[3]{2}$$

2. Simplify the first radical

$$2\sqrt[3]{2}$$

Example

Problem?

$$\sqrt[3]{24x^2y^5}$$

Is there another exponent that is too big?

$$\sqrt[3]{2^3 3x^2y^5}$$

Write this as a product of two radicals :

$$\sqrt[3]{2^3y^3}\,\sqrt[3]{3x^2y^2}$$

Simplify the first radical :

$$2y\sqrt[3]{3x^2y^2}$$

Simplifying when a Denominator Contains a Single Radical of Index "n"

1. Simplify the top and bottom separately to get rid of exponents under the radical that are too big
2. Multiply the whole fraction by a special kind of "1" where 1 is in the form of: $\dfrac{\sqrt[n]{m}}{\sqrt[n]{m}}$

   $\text{and } m \text{ is the product of all the factors required to make every exponent in the radicand be equal to } "n"$

3. Simplify to eliminate the radical in the denominator

Example

$$\frac{3}{\sqrt[5]{4x^3y^6}} = \frac{3}{\sqrt[5]{2^2x^3y^6}} = \frac{3}{\sqrt[5]{y^5}\sqrt[5]{2^2x^3y}} = \frac{3}{y\sqrt[5]{2^2x^3y}}$$

$$= \frac{3}{y\sqrt[5]{2^2x^3y}} \cdot \frac{\sqrt[5]{2^3x^2y^4}}{\sqrt[5]{2^3x^2y^4}} = \frac{3\sqrt[5]{2^3x^2y^4}}{y\sqrt[5]{2^5x^5y^5}} = \frac{3\sqrt[5]{2^3x^2y^4}}{2xy^2}$$

$$= \frac{3\sqrt[5]{8x^2y^4}}{2xy^2}$$

Simplifying when Radicand is a Fraction

1. Use the quotient rule to **write** the **single radical as** a quotient of **two radicals**
2. **Use** the **rules already learned** for simplifying when there is a radical in a denominator

Example

$$\sqrt[5]{\frac{3}{4}} = \frac{\sqrt[5]{3}}{\sqrt[5]{4}} = \frac{\sqrt[5]{3}}{\sqrt[5]{2^2}} = \frac{\sqrt[5]{3}}{\sqrt[5]{2^2}} \cdot \frac{\sqrt[5]{2^3}}{\sqrt[5]{2^3}} = \frac{\sqrt[5]{3 \cdot 2^3}}{\sqrt[5]{2^5}}$$

$$= \frac{\sqrt[5]{24}}{2}$$

Simplifying when All Exponents in Radicand Share a Common Factor with Index

1. **Divide out** the **common factor** from the index and all exponents

$$\sqrt[6]{2^4 3^6 x^8 y^2}$$

All exponents in radicand and index share what factor? 2

Dividing all exponents in and index by 2 gives :

$$\sqrt[3]{2^2 3^3 x^4 y} = \sqrt[3]{3^3 x^3}\sqrt[3]{2^2 xy} = \boxed{3x\sqrt[3]{4xy}}$$

Problem?

Simplifying Expressions Involving Products and/or Quotients of Radicals with the Same Index

- Use the product and quotient rules to **combine** everything **under** a **single radical**
- **Simplify** the single radical **by procedures previously discussed**

Example

$$\frac{\sqrt[4]{ab^3}\,\sqrt[4]{ab}}{\sqrt[4]{a^3b^3}} = \sqrt[4]{\frac{a^2b^4}{a^3b^3}} = \sqrt[4]{\frac{b}{a}} = \frac{\sqrt[4]{b}}{\sqrt[4]{a}} = \frac{\sqrt[4]{b}}{\sqrt[4]{a}}\,\frac{\sqrt[4]{a^3}}{\sqrt[4]{a^3}}$$

$$= \frac{\sqrt[4]{a^3b}}{\sqrt[4]{a^4}} = \frac{\sqrt[4]{a^3b}}{a}$$

Adding and Subtracting Radicals

- **Addition and subtraction of radicals** can always be indicated, but can be **simplified into** a **single radical <u>only</u> when** the radicals are **"like radicals"**
- **"Like Radicals"** are radicals that have exactly the **same index and radicand**, but may have different coefficients

Which are like radicals? $3\sqrt[4]{5},\ 4\sqrt{5}, -2\sqrt[4]{5} \text{ and } 3\sqrt[3]{5}$

- When **"like radicals"** are **added or subtracted**, the **result is** a **"like radical"** with coefficient equal to the sum or difference of the coefficients

$$3\sqrt[4]{5} + 2\sqrt[4]{5} = 5\sqrt[4]{5}$$

$$-2\sqrt[4]{5} + 3\sqrt[3]{5} = \text{Okay as is - can't combine unlike radicals}$$

Note Concerning Adding and Subtracting Radicals

- When **addition or subtraction of radicals** is indicated you must **first simplify all radicals** because some radicals that do not appear to be like radicals become like radicals when simplified

Example

Not like terms (yet)

$$\sqrt[3]{128} - 5\sqrt[3]{2} + 2\sqrt[3]{16}$$

Simplify individual radicals :

$$= \sqrt[3]{2^7} - 5\sqrt[3]{2} + 2\sqrt[3]{2^4}$$

$$= \sqrt[3]{2^3 2^3}\ \sqrt[3]{2} - 5\sqrt[3]{2} + 2\sqrt[3]{2^3}\ \sqrt[3]{2}$$

$$= 2 \cdot 2\sqrt[3]{2} - 5\sqrt[3]{2} + 2 \cdot 2\sqrt[3]{2}$$

All like radicals :

$$= 4\sqrt[3]{2} - 5\sqrt[3]{2} + 4\sqrt[3]{2}$$

$$= 3\sqrt[3]{2}$$

Simplifying when there is a Single Radical Term in a Denominator

1. **Simplify** the radical in the **denominator**
2. If the denominator still contains a radical, **multiply** the fraction **by** "1" where "1" is in the form of a **"special radical" over itself**
3. The "special radical" is one that contains the factors necessary to **make** the denominator radical factors have **exponents equal** to **index**
4. **Simplify** radical in **denominator** to eliminate it

Example

$$\frac{\sqrt[3]{2}}{\sqrt[3]{9x}}$$ Simplify denominator:

$$\frac{\sqrt[3]{2}}{\sqrt[3]{3^2 x}} \cdot 1$$ Multiply by special "1":

$$\frac{\sqrt[3]{2}}{\sqrt[3]{3^2 x}} \cdot \frac{\sqrt[3]{3x^2}}{\sqrt[3]{3x^2}}$$ Use product rule:

$$\frac{\sqrt[3]{2 \cdot 3x^2}}{\sqrt[3]{3^3 x^3}}$$ Simplify denominator:

$$\frac{\sqrt[3]{6x^2}}{3x}$$

Simplifying to Get Rid of a Binomial **Denominator** that **Contains One or Two Square Root** Radicals

1. **Simplify** the radical(s) in the **denominator**
2. If the denominator still contains a radical, **multiply** the fraction **by** “1” where “1” is in the form of a **“special binomial radical” over itself**
3. The “special binomial radical” is the ***conjugate*** of the denominator (***same terms – opposite sign***)
4. **Complete multiplication** (the denominator will contain no radical)

Example

$$\frac{\sqrt{5}}{\sqrt{3}+\sqrt{2}}$$

Radical in denominator doesn't need simplifying

Multiply fraction by special one:

$$\frac{\sqrt{5}}{\sqrt{3}+\sqrt{2}} \cdot \frac{\sqrt{3}-\sqrt{2}}{\sqrt{3}-\sqrt{2}}$$

Distribute on top:

FOIL on bottom:

$$\frac{\sqrt{15}-\sqrt{10}}{\sqrt{9}-\sqrt{4}}$$

Simplify bottom:

$$\frac{\sqrt{15}-\sqrt{10}}{3-2}$$

$$\sqrt{15}-\sqrt{10}$$

Problem 14

- Simplify:

$$\sqrt[4]{x^8 y^7 z^9}$$

- Answer: $x^2 yz^2 \sqrt[4]{y^3 z}$

Problem 15

- Simplify:

$$\sqrt[3]{\frac{16}{x^2}}$$

- Answer: $\dfrac{2\sqrt[3]{2x}}{x}$

Problem 16

- Simplify:

$$\sqrt[6]{x^3}$$

- Answer:

$$\sqrt{x}$$

Problem 17

- Simplify:

$$\frac{1}{2-\sqrt{x}}$$

- Answer: $$\frac{2+\sqrt{x}}{4-x}$$

Problem 18

- Simplify:

$$\sqrt[3]{16} - \sqrt[3]{2}$$

- Answer:

$$\sqrt[3]{2}$$